

# On the regular $k$ -independence number of graphs \*

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## Abstract

The *regular independence number*, introduced by Albertson and Boutin in 1990, is the maximum cardinality of an independent set of  $G$  in which all vertices have equal degree in  $G$ . Recently, Caro, Hansberg and Pepper introduced the concept of regular  $k$ -independence number, which is a natural generalization of the regular independence number. A  *$k$ -independent set* is a set of vertices whose induced subgraph has maximum degree at most  $k$ . The *regular  $k$ -independence number* of  $G$ , denoted by  $\alpha_{k-reg}(G)$ , is defined as the maximum cardinality of a  $k$ -independent set of  $G$  in which all vertices have equal degree in  $G$ . In this paper, the exact values of the regular  $k$ -independence numbers of some special graphs are obtained. We also get some lower and upper bounds for the regular  $k$ -independence number of trees with given diameter, and the lower bounds for the regular  $k$ -independence number of line graphs. For a simple graph  $G$  of order  $n$ , we show that  $1 \leq \alpha_{k-reg}(G) \leq n$  and characterize the extremal graphs. The Nordhaus-Gaddum-type results for the regular  $k$ -independence number of graphs are also obtained.

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## 1 Introduction

Graphs considered in this paper are undirected, finite and simple. We refer to [3] for undefined notations and terminology. In particular, we use  $L(G)$ ,  $\bar{G}$ ,  $\Delta(G)$  and  $\delta(G)$  to denote the line graph, the complementary graph, the maximum degree and minimum degree of a graph  $G$ , respectively. If  $X \subseteq V(G)$  or  $X \subseteq E(G)$ , then  $G[X]$  is the subgraph of  $G$  induced by  $X$ . For integers  $i, j \geq 0$ , let  $D_i(G)$  denote the set of degree  $i$  vertices of  $G$ , and  $D_{\geq i}(G) = \bigcup_{j \geq i} D_j(G)$ . A subset  $X \subseteq V(G)$  is *regular* if for some  $i$  with  $\delta(G) \leq i \leq \Delta(G)$ ,  $X \subseteq D_i(G)$ ; and is independent if  $\Delta(G[X]) = 0$ . An independent set  $X$  of  $G$  is a *regular independent set* if  $X$  is also regular.

The *regular independence number*, denoted  $\alpha_{reg}(G)$  and introduced by Albertson and Boutin [1] in 1990, is defined to be the maximum cardinality of an independent set of  $G$  in which all vertices have equal degree in  $G$ . The parameter  $\alpha_{reg}(G)$  is closely related to fair domination number  $fd(G)$  introduced in [6]. A *fair dominating set* is a set  $S \subseteq V(G)$  such that all vertices  $v \in V(G) \setminus S$  have exactly the same non-zero number of neighbors in  $S$ . The *fair domination number*  $fd(G)$  is the cardinality of a minimum fair dominating set of  $G$ . By definition, if  $\delta(G) \geq 1$  and  $R$  is a maximum regular independent set of  $G$ , then  $V(G) \setminus R$  is a fair dominating set of  $G$ . A vertex subset  $S \subseteq V(G)$  of  $G$  is  *$k$ -independent set* if  $\Delta(G[S]) \leq k$ . The  *$k$ -independence number*, denoted  $\alpha_k(G)$ , as the maximum cardinality of a  $k$ -independent set. There have been quite a few studies on  $k$ -independent sets, as seen in [9, 5, 10], among others. For  $k$ -independent set and  $k$ -independence number, Chellali, Favaron, Hansberg, and Volkmann published a survey paper on this subject; see [4].

Recently, Caro, Hansberg and Pepper [7] introduced the concept of regular  $k$ -independence number, which naturally generalizes both the regular independence number and the  $k$ -independence number. The *regular  $k$ -independence number* of a graph  $G$ , denoted  $\alpha_{k-reg}(G)$ , is defined to be the maximum cardinality of a regular  $k$ -independent set of  $G$ . More precisely, for nonnegative integers  $k$  and  $j$ , we define  $\alpha_{k,j}(G) = \max\{|X| : X \text{ is a } k\text{-independent set of } G \text{ and } X \subseteq D_j(G)\}$ . It follows by definition that,

$$\alpha_{k-reg}(G) = \max\{\alpha_{k,j}(G), \delta(G) \leq j \leq \Delta(G)\}. \quad (1.1)$$

When  $k = 0$ ,  $\alpha_{0-reg}(G) = \alpha_{reg}(G)$  and for regular graphs,  $\alpha_{reg}(G) = \alpha(G)$  and  $\alpha_{k-reg}(G) = \alpha_k(G)$ .

For each integer  $i \geq 0$ , define  $n_i(G) = |D_i(G)|$ . We often write  $n_i$  for  $n_i(G)$  when the

graph  $G$  is understood from the context. The *repetition number* of  $G$ , denoted  $rep(G)$ , was introduced in [8] and defined as the maximum number of vertices with equal degree in  $G$ . Thus

$$rep(G) = \max\{|D_i(G)| : \delta(G) \leq i \leq \Delta(G)\}. \quad (1.2)$$

The notation of  $\chi_k(G)$  is the *k-chromatic number* of  $G$ , defined as the minimum number of colors needed to color the vertices of the graphs  $G$  such that the graphs induced by the vertices of each color class have maximum degree at most  $k$ . Note that  $\chi_0(G)$  is the classic chromatic number  $\chi(G)$ .

In [7], Caro, Hansberg and Pepper investigated the regular  $k$ -independence number of trees and forests, and they generalized and extended the results of Albertson and Boutin [1] in to  $\alpha_{k-reg}(G)$ . They presented a lower bound on  $\alpha_{k-reg}(G)$  for  $k$ -trees and gave analogous results for  $k$ -degenerate graphs and some specific results about planar graphs, and then gave lower bounds on  $\alpha_{2-reg}(G)$  for planar and outerplanar graphs. The authors also analyzed complexity issues of regular  $k$ -independence.

This paper is organized as follows. In Section 2, the exact values of the regular  $k$ -independence numbers of complete graphs, complete multipartite graphs, paths, cycles and stars are determined. Sharp bounds for the regular  $k$ -independence number of trees with given diameter are obtained in Section 3. In Section 4, we obtain the lower bounds for the regular  $k$ -independence number of general  $m$ -vertex line graphs. For some families of sparse graphs such as trees, maximal outerplanar graphs and triangulations, we present lower bounds for the regular  $k$ -independence number of their line graphs, which improve several former results. For a simple graph  $G$  of order  $n$ , we show that  $1 \leq \alpha_{k-reg}(G) \leq n$ , and characterize all extremal graphs in Section 5.

Let  $\mathcal{G}(n)$  denote the class of simple graphs of order  $n$  ( $n \geq 2$ ). For  $G \in \mathcal{G}(n)$ ,  $\bar{G}$  denotes the complement of  $G$ . Give a graph parameter  $f(G)$  and a positive integer  $n$ , the *Nordhaus-Gaddum(N-G) Problem* is to determine sharp bounds for both  $f(G) + f(\bar{G})$  and  $f(G) \cdot f(\bar{G})$ , as  $G$  ranges over the class  $\mathcal{G}(n)$ , and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention, as seen in the survey paper [2] by Aouchiche and Hansen. The Nordhaus-Gaddum-type problem on the regular  $k$ -independence number of graphs is studied in Section 6.

## 2 Results for some special graphs

In this section, we will determine the regular  $k$ -independence numbers in several special families of graphs. Throughout this section,  $n > 0$  denotes an integer.

**Proposition 2.1** *Let  $K_n$  be a complete graph of order  $n$ . Then*

$$\alpha_{k-reg}(K_n) = \begin{cases} i+1, & \text{if } k = i \ (0 \leq i \leq n-1); \\ n, & \text{if } k \geq n. \end{cases}$$

*Proof.* Let  $K_n$  be a complete graph with  $n$  vertices. Since  $K_n$  is a regular graph, it follows that  $\alpha_{k-reg}(K_n) = \alpha_k(K_n)$ . By the definition of  $\alpha_k(K_n)$ , if  $k = i$  for some  $i$  with  $0 \leq i \leq n-1$ , then every  $i+1$  subset of vertices is a maximum regular  $k$ -independent set of  $K_n$ ; if  $k \geq n$ , then  $V(K_n)$  is a maximum regular  $k$ -independent set of  $K_n$ . This proves the proposition.  $\blacksquare$

**Proposition 2.2** *Let  $K_{r_1, r_2, \dots, r_n}$  be a complete  $n$ -partite graph.*

(1) *If  $r_1 = r_2 = \dots = r_n = a$ , then*

$$\alpha_{k-reg}(K_{r_1, r_2, \dots, r_n}) = \begin{cases} ia, & \text{if } (i-1)a \leq k < ia \ (1 \leq i \leq n); \\ na, & \text{if } k \geq na. \end{cases}$$

(2) *If  $r_1 < r_2 < \dots < r_n$ , then  $\alpha_{k-reg}(K_{r_1, r_2, \dots, r_n}) = r_n$  for  $k \geq 0$ .*

(3) *If  $r_1 < \dots < r_i = r_{i+1} = \dots = r_j < r_{j+1} < \dots < r_n$  ( $i < j$ ), then*

$$\alpha_{k-reg}(K_{r_1, r_2, \dots, r_n}) = \begin{cases} \max\{mr_i, r_n\}, & \text{if } (m-1)r_i \leq k < mr_i \ (1 \leq m \leq (j-i)); \\ \max\{(j-i)r_i, r_n\}, & \text{if } k \geq (j-i)r_i. \end{cases}$$

*Proof.* Let  $G = K_{r_1, r_2, \dots, r_n}$  with partite sets  $V_1, V_2, \dots, V_n$  such that  $|V_j| = r_j$ ,  $1 \leq j \leq n$ .

(1) Assume that  $r_1 = r_2 = \dots = r_n = a$ . Then  $K_{r_1, r_2, \dots, r_n}$  is a regular graph, and so  $\alpha_{k-reg}(K_{r_1, r_2, \dots, r_n}) = \alpha_k(K_{r_1, r_2, \dots, r_n})$ . For each  $i \in \{1, 2, \dots, n\}$  with  $(i-1)a \leq k < ia$ , any union of  $i$  of the partite sets is a maximum regular  $k$ -independent set; and for  $k \geq na$ ,  $V(G)$  is the only maximum regular  $k$ -independent set. This justifies (1).

(2) Assume that  $r_1 < r_2 < \dots < r_n$ . By the definition of regular independent sets and since the  $r_i$ 's are mutually distinct, a vertex subset  $X$  of  $G$  is a regular independent set if and only if  $X \subseteq V_t$  for some  $t$  with  $1 \leq t \leq n$ . It follows that  $V_n$  is the only maximum regular  $k$ -independent set of  $G$ . Thus  $\alpha_{k-reg}(K_{r_1, r_2, \dots, r_n}) = r_n$  for  $k \geq 0$ .

(3) Assume that for some integers  $1 \leq i < j \leq n$ ,  $r_1 < \dots < r_i = r_{i+1} = \dots = r_j < r_{j+1} < \dots < r_n$ . For a fixed integer  $k$ , let  $X$  be a maximum regular  $k$ -independent set of  $G$ . By assumption of (3) and by the definition of regular independent sets, we note that either  $X \subseteq V_t$  for some  $t$  with  $1 \leq t \leq n$  or  $X \subseteq \bigcup_{t=i}^j V_t$ . If for an integer  $m$  with  $1 \leq m \leq (j-i)$ , we have  $(m-1)r_i \leq k < mr_i$ , then either  $X \subseteq \bigcup_{t=i}^j V_t$  with  $|X| = mr_i$  or  $X = V_n$ , whence  $|X| = \max\{mr_i, r_n\}$ . If  $k \geq (j-i)r_i$ , then either  $X = \bigcup_{t=i}^j V_t$  or  $X = V_n$ , whence  $|X| = \max\{(j-i)r_i, r_n\}$ . This verifies (3).  $\blacksquare$

**Proposition 2.3** Let  $m \geq 2$  be an integer,  $i \in \{0, 1, 2\}$ ,  $P_n$  be a path of order  $n$ , where  $n = 3(m - 2) + 2 + i$ . Each of the following holds.

(1) If  $m \geq 3$ , then

$$\alpha_{k-reg}(P_n) = \begin{cases} \lceil \frac{n-2}{2} \rceil, & \text{if } k = 0; \\ n - m, & \text{if } k = 1; \\ n - 2, & \text{if } k \geq 2. \end{cases}$$

(2) If  $m = 2$ , then

$$\alpha_{k-reg}(P_2) = \begin{cases} 1, & \text{if } k = 0; \\ 2, & \text{if } k \geq 1. \end{cases}$$

and

$$\alpha_{k-reg}(P_3) = \alpha_{k-reg}(P_4) = 2$$

for  $k \geq 0$ .

*Proof.* As the proof for (2) is straightforward, we only need to show the validity of (1). Assume that  $m \geq 3$ . Then  $n \geq 5$ , and  $D_2(P_n) = n - 2$ . If  $k = 0$ , then  $D_2(P_n)$  contains an independent subset  $W_0$  with  $|W_0| = \lceil \frac{n-2}{2} \rceil$ . Likewise, if  $k = 1$ , then  $D_2(P_n)$  contains a 1-independent set  $W_1$  consisting  $i$  isolated vertices and the vertex set of a matching with  $\frac{n-2-i}{3}$  edges. It is routine to show that  $W_1$  is a maximum 1-independent set, and so  $\alpha_{1-reg}(P_n) = 2(\frac{n-2-i}{3}) + i = 2(m - 2) + i = 2m - 4 + i = n - m$ . If  $k \geq 2$ , then  $D_2(P_n)$  is a maximum regular  $k$ -independent set, and so  $\alpha_{k-reg}(P_n) = n - 2$ . ■

**Proposition 2.4** Let  $C_n$  be a cycle of order  $n$ . Then

$$\alpha_{k-reg}(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } k = 0; \\ 2a, & \text{if } k = 1 \text{ and } n = 3a \text{ or } n = 3a + 1; \\ 2a + 1, & \text{if } k = 1 \text{ and } n = 3a + 2; \\ n, & \text{if } k \geq 2. \end{cases}$$

*Proof.* Denote  $C_n = v_1 v_2 \cdots v_n v_1$ . As  $C_n$  is regular, any  $k$ -independent set of  $C_n$  is also a regular  $k$ -independent set. If  $k = 0$ , then  $\{v_{2i+1} : 0 \leq i \leq \frac{n}{2} - 1\}$  is a maximum regular independent set of  $C_n$ . Hence  $\alpha_{0-reg}(C_n) = \lfloor \frac{n}{2} \rfloor$ . If  $k = 1$ , then  $V(C_n) - \{v_{3i} : 1 \leq i \leq \frac{n}{3}\}$  (if  $n \equiv 0 \pmod{3}$ ) or  $V(C_n) - (\{v_{3i} : 1 \leq i \leq \lfloor \frac{n}{3} \rfloor\} \cup \{v_n\})$  (if  $n \equiv 1$  or  $n \equiv 2 \pmod{3}$ ) is a maximum regular  $k$ -independent set of  $C_n$ . If  $k \geq 2$ ,  $V(C_n)$  is a maximum regular  $k$ -independent set. This justifies the proposition. ■

**Proposition 2.5** Let  $S_{1,n-1}$  be a star of order  $n$ . Then  $\alpha_{k-reg}(S_{1,n-1}) = n - 1$  for  $k \geq 0$ .

*Proof.* Since  $S_{1,n-1}$  has  $n$  vertices, there are  $n-1$  vertices of degree 1 in  $S_{1,n-1}$ . When  $k \geq 0$ , the subgraph induced by all the vertices of degree 1 in  $S_{1,n-1}$  is a regular  $k$ -independent set. By the definition of the regular  $k$ -independence number,  $\alpha_{k-reg}(S_{1,n-1}) = n-1$ . ■

### 3 Results for trees with given diameter

Caro, Hansberg and Pepper [7] generalized and extended the result that  $\alpha_{reg}(T) \geq \frac{n+2}{4}$  for any tree  $T$ , obtained by Albertson and Boutin in [1]. They showed that for every tree  $T$  on  $n \geq 2$  vertices,  $\alpha_{k-reg}(T) \geq \frac{2(n+2)}{7}$  for  $k = 1$  and  $\alpha_{k-reg}(T) \geq \frac{(n+2)}{3}$  for  $k \geq 2$ . In this section, we improve the bound of  $\alpha_{k-reg}(T)$  for  $k \geq 2$  by considering the diameter of a tree. Throughout this section, let  $n \geq 8$ ,  $t > 0$  and  $k \geq 2$  be integers, and  $T_{n,t}$  denote the family of trees with order  $n$  and diameter  $n-t$ . For notational convenience, when it is clear from the context, we also use  $T_{n,t}$  to denote a member in this family. Thus  $T_{n,1}$  is a path with  $n$  vertices and  $T_{n,n-2}$  is a star with  $n$  vertices. As the regular  $k$ -independence number of paths and stars are determined in the section above, we always assume that  $2 \leq t \leq n-3$  in this section. If  $G = T_{n,t}$ , then for each  $i \geq 1$ , let  $n_i = |D_i(G)|$  and  $N_3 = |D_{\geq 3}(G)|$ . In a graph  $H$ , an *elementary subdivision* of an edge  $uv$  is the operation of replacing  $uv$  with a path  $uvw$  through a new vertex  $w$ . A *subdivision* of  $H$  is the graph obtained by a finite sequence of elementary subdivisions on  $H$ . As usual, a *leaf* of a tree is a vertex of degree 1 in the tree. The main purpose of this section is to investigate the regular independence number for trees with given diameters. We start with lemmas and examples.

**Lemma 3.1** *Let  $T$  be a tree on  $n \geq 2$  vertices. Each of the following holds.*

- (i)  $|D_1(T)| = \sum_{v \in D_{\geq 3}(G)} (d_T(v) - 2) + 2$ .
- (ii) If  $T = T_{n,t}$ , then  $N_3 = |D_{\geq 3}(T_{n,t})| = n - n_1 - n_2$ . and  $n_1 \geq N_3 + 2$ .

*Proof.* We outline our proofs. Lemma 3.1(i) holds if  $|V(T)| = 2$ , and so it can be justified by induction on  $|V(T)|$ , by considering  $T - v$  for some  $v \in D_1(T)$  in the inductive step. Lemma 3.1(ii) follows from the definitions. ■

**Example 3.2** *Let  $\ell_1 \geq \ell_2 \cdots \ell_r \geq 1$  and  $r \geq 3$  be integers, and let  $K_{1,r}$  denote the tree with a vertex  $v_0$  of degree  $r$  and  $D_1(K_{1,r}) = \{v_1, v_2, \dots, v_r\}$ .*

(i) *Define  $K_{1,r}(\ell_1, \ell_2, \dots, \ell_r)$  to be the graph obtained from  $K_{1,r}$  by replacing each edge  $v_0 v_i$  by a  $(v_0, v_i)$ -path of order  $\ell_i$ , for each  $i$  with  $1 \leq i \leq r$ . When  $\ell_3 = \ell_{r-1}$ , we also use  $K_{1,r}(\ell_1, \ell_2, \ell_3^{r-3}, \ell_r)$  for  $K_{1,r}(\ell_1, \ell_2, \dots, \ell_r)$ . Let  $T = K_{1,r}(\ell_1, \ell_2, \ell_3^{r-3}, \ell_r)$ . It is elementary to compute that  $T = T_{n,t}$  with  $n = 1 + \sum_{i=1}^r (\ell_i - 1)$  and  $n - t = \ell_1 + \ell_2 - 2$ . If  $k \geq 2$ , then*

$$\alpha_{k-reg}(T) = \begin{cases} r & \text{if } r > \frac{n-1}{2} \\ n - r - 1 & \text{if } r \leq \frac{n-1}{2} \end{cases}.$$

(ii) Let  $T$  be a tree with  $|D_{\geq 3}(T)| \geq 2$ . Assume that  $z, z' \in D_{\geq 3}(T)$  and  $T_1, T_2$  be the two subtrees of  $T$  such that  $T = T_1 \cup T_2$ ,  $V(T_1) \cap V(T_2) = \{z'\}$ ,  $z \in V(T_1)$  and  $z' \in D_2(T_1)$ . View  $T_2'$  is a copy of  $T_2$  but vertex disjoint from  $V(T_1)$ . Obtain a new tree  $T'$  from the vertex disjoint union of  $T_1$  and  $T_2'$  by identifying  $z \in V(T_1)$  and  $z' \in V(T_2')$ . We use  $O_{z' \rightarrow z}$  to denote this operation and write  $T' = O_{z' \rightarrow z}(T)$ , and use  $O_{z' \leftarrow z}$  to denote the reverse operation. Hence  $T = O_{z' \leftarrow z}(T')$ . By definition and by Lemma 3.1,  $|V(T)| = |V(T')|$ ,  $|D_1(T)| = |D_1(T')|$  and  $|D_{\geq 3}(T)| - 1 = |D_{\geq 3}(T')|$ . For a fixed  $z$ , define relation  $T \sim T'$  if and only if for some  $z'$ ,  $T' = O_{z' \rightarrow z}(T)$ . Then  $\sim$  is an equivalence relation on the set of all trees with the same number of vertices and same number of leaves.

(iii) Let  $T$  be a given tree  $T$  with  $|D_{\geq 3}(T)| \geq 1$ , and let  $z \in D_{\geq 3}(T)$  be a fixed vertex. Define  $\mathcal{F}(T, z)$  to be equivalence class containing  $T$  under the relation  $\sim$  defined in (ii) above. By definition,  $T \in \mathcal{F}(T, z)$ .

(iv) Suppose that  $n$  and  $t$  are integers with  $2 \leq t \leq n - 3$ . Let  $h = \lfloor \frac{n-t}{2} \rfloor$ . Since  $t \geq 2$ , we can write  $n - 1 = qh + r$  for some integers  $q \geq 2$  and  $1 \leq r \leq h$ . Define

$$T(n, t) = \begin{cases} K_{1, q+1}(h+1, h+1, (h+1)^{q-2}, r+1) & \text{if } n-t \equiv 0 \pmod{2} \\ K_{1, q+1}(h+2, h+1, (h+1)^{q-2}, r+1) & \text{if } n-t \equiv 1 \pmod{2} \end{cases},$$

and let  $z_0$  be the only vertex in  $T(n, t)$  with degree  $q+1$ . By Example 3.2(iii), for each  $T \in \mathcal{F}(T(n, t), z_0)$ ,  $|D_1(T)| = q+1$ . By definition, the diameter of  $T(n, t)$  is  $n-t$ . Direct computation yields that  $|D_1(T(n, t))| = q+1$  and  $|D_2(T(n, t))| = n-q-2$ .

For integers  $n > t \geq 2$  with  $t \leq n - 3$ , define

$$f(n, t) = \begin{cases} \left\lceil \frac{2(t-1)}{n-t} \right\rceil + 2 & \text{if } n-t \equiv 0 \pmod{2} \\ \left\lceil \frac{2(t-1)}{n-t-1} \right\rceil + 2 & \text{if } n-t \equiv 1 \pmod{2} \end{cases}, \quad (3.3)$$

**Lemma 3.3** Suppose that  $T = T_{n,t}$  with  $2 \leq t \leq n - 3$ . Let  $P = v_1 v_2, \dots, v_{n-t+1}$  be a longest path in  $T(n, t)$  (as defined in Example 3.2(iv)),  $h = \lfloor \frac{n-t}{2} \rfloor$ , and  $z_0 = v_h$ ,  $z_0 = v_{h+1}$ . Express  $n - 1 = qh + r$  for some integers  $q \geq 2$  and  $1 \leq r \leq h$ . Then each of the following holds.

(i)  $n_1 = |D_1(T)| \geq f(n, t)$ .

(ii) Equality in (i) holds if and only if both  $q+1 = f(n, t)$  and  $T \in \mathcal{F}(T(n, t), z_0)$ .

*Proof.*

Since  $T$  is connected and since  $N_3 > 0$ , there must be a  $j_0$  with  $v_{j_0} \in V(P) \cap D_{\geq 3}(G)$ . Without lose of generality, we assume that  $1 < j_0 < n - t + 1$  such that  $|\lceil \frac{n-t+1}{2} \rceil - j_0|$  is minimized. By symmetry, we may assume that  $1 < j_0 \leq \lceil \frac{n-t+1}{2} \rceil = h+1$ . We shall argue by induction on  $N_3$ . Since  $t \leq n - 3$ , we have  $N_3 > 0$ .

Suppose that  $N_3 = 1$ . Then for any  $w \in D_1(T) - \{v_1, v_{n-t+1}\}$ , there exists a unique  $(w, v_{j_0})$ -path  $P_w$  in  $T$  such that  $V(P_w) \cap V(P) = \{v_{j_0}\}$ .

Assume first that  $n - t \equiv 0 \pmod{2}$ , and so  $n - t = 2h$ . Since the diameter of  $T$  is  $n - t$ , and since  $j_0 \leq h + 1$  for any  $w \in D_1(T) - \{v_1, v_{n-t+1}\}$ ,  $|E(P_w)| \leq j_0 - 1 \leq h$ . It follows that

$$\begin{aligned} n - 1 &= |V(T) - \{v_{j_0}\}| = |V(P) - \{v_{j_0}\}| + \sum_{w \in D_1(T) - \{v_1, v_{n-t+1}\}} |V(P_w - v_{j_0})| \quad (3.4) \\ &\leq |V(P)| - 1 + (n_1 - 2)(j_0 - 1) \leq n - t + (n_1 - 2)h, \end{aligned}$$

and so  $n_1 \geq f(n, t)$ . Assume that we have  $n_1 = f(n, t)$ . Then, if  $h$  divides  $t - 1$ , then every inequality in (3.4) must be an equality; and if  $h$  does not divide  $t - 1$ , then for some integer  $r'$  with  $0 < r' < h$ ,  $n - 1 = n - t + (n_1 - 2)h - r'$ . It follows that  $j_0 = h + 1 = |V(P_w)|$ , for all but at most one  $w \in D_1(T) - \{v_1, v_{n-t+1}\}$ . Since  $n - 1 = qh + r$  with  $1 \leq r \leq h$ , we have  $n_1 = q + 1$ . As  $N_3 = 1$ ,  $T$  must be a subdivision of  $K_{1, n_1}$ , and so  $T = K_{1, n_1}(h + 1, h + 1, (h + 1)^{q-2}, r + 1) = T(n, t)$ .

The proof for the case when  $n - t \equiv 1 \pmod{2}$  is similar, using  $n - t = 2h + 1$  and  $(n_1 - 2)h + n - t \geq n - 1$  instead, and so it is omitted.

We now assume that  $N_3 > 1$ , and that Lemma 3.3 holds for smaller values of  $N_3$ . Since  $N_3 \geq 2$ , there exists a  $w \in D_{\geq 3}(T) - \{v_{j_0}\}$ . Let  $T' = O_{w \rightarrow v_{j_0}}$ . By Example 3.2,  $|D_1(T')| = |D_1(T)| = n_1$ . As  $j_0$  is so chosen that  $|\lceil \frac{n-t+1}{2} \rceil - j_0|$  is minimized, the diameter of  $T'$  is also  $n - t$ . However,  $D_{\geq 3}(T') = D_{\geq 3}(T) - \{w\}$ . By induction,

$$n_1 = |D_1(T)| = |D_1(T')| \geq f(n, t).$$

If equality holds, then by induction,  $T' \in \mathcal{F}(T(n, t), v_{j_0})$ , where  $j_0 = h + 1$ . This complete the proof of the lemma.  $\blacksquare$

**Lemma 3.4** *Suppose that  $k \geq 2$  and  $T = T_{n, t}$  with  $2 \leq t \leq n - 3$ . Then  $\alpha_{k-reg}(T) = \max\{|D_1(T)|, |D_2(T)|\}$ .*

*Proof.* Since  $k \geq 2$ , both  $D_1(T)$  and  $D_2(T)$  are regular 2-independent sets of  $T$ . Therefore,  $\alpha_{k-reg}(T) \geq \max\{|D_1(T)|, |D_2(T)|\}$ . If  $X$  is a maximum  $k$ -independent set of  $T$ , then for some  $i$ ,  $X \subseteq D_i(T)$ , and so  $\alpha_{k-reg}(T) = |X| \leq |D_i(T)| \leq \max\{|D_i(T)| : i \geq 1\}$ . By Lemma 3.1(ii),  $|D_1(T)| \geq N_3 + 2 > N_3 = \sum_{j \geq 3} |D_j(T)|$ . This implies that  $\max\{|D_1(T)|, |D_2(T)|\} \leq \alpha_{k-reg}(T) \leq \max\{|D_i(T)| : i \geq 1\} \leq \max\{|D_1(T)|, |D_2(T)|\}$ .  $\blacksquare$

**Theorem 3.5** *Let  $k \geq 2$  be an integer and  $T_{n, t}$  be a tree with order  $n \geq 8$  and diameter  $n - t$  with  $2 \leq t \leq n - 3$ .*

(i) *If  $2 \leq t \leq \frac{n-1}{3}$ , then*

$$n - 2t \leq \alpha_{k-reg}(T_{n, t}) \leq n - 4. \quad (3.5)$$

(ii) *If  $\frac{n}{3} \leq t \leq n - 5$  ( $n \geq 8$ ), then*

$$\frac{n+2}{3} \leq \alpha_{k-reg}(T_{n, t}) \leq \max\{n - f(n, t) - 1, t + 1\}. \quad (3.6)$$



(iii) If  $t = n - 4$ , then

$$\left\lceil \frac{n-1}{2} \right\rceil \leq \alpha_{k-reg}(T_{n,t}) \leq t+1. \quad (3.7)$$

(iv) If  $t = n - 3$ , then

$$\alpha_{k-reg}(T_{n,t}) = t+1. \quad (3.8)$$

*Proof.* Since  $k \geq 2$ , both  $D_1(T_{n,t})$  and  $D_2(T_{n,t})$  are regular  $k$ -independent set. By the definition of  $T_{n,t}$ ,  $n_1 \leq t+1$ . By Lemma 3.3 and by  $N_3 \geq 1$ ,

$$n_2 \leq n - f(n, t) - 1,$$

where equality holds if and only if  $N_3 = 1$  and  $n_1 = f(n, t)$ . By Lemma 3.4,

$$\alpha_{k-reg}(T_{n,t}) = \max\{n_1, n_2\} \leq \max\{n - f(n, t) - 1, t+1\}. \quad (3.9)$$

(i) Suppose  $2 \leq t \leq \frac{n-1}{3}$ . Since  $n_2 = n - n_1 - N_3$  and  $n_1 \geq N_3 + 2$ , it follows that  $n_2 \geq n - 2n_1 + 2$ . Since  $\text{diam}(T_{n,t}) = n - t$ , we have  $n_1 \leq t+1$ . If  $t \leq \frac{n-1}{3}$ , then  $n_2 \geq n_1$ . Thus  $D_2(G)$  is a maximum regular  $k$ -independent set and so  $\alpha_{k-reg}(T_{n,t}) = n_2$ . By definition,  $T_{n,t}$  contains a path  $P = v_1 v_2 \cdots v_{n-t+1}$ . As the remaining  $t-1$  vertices in  $V(T_{n,t}) - V(P)$  are adjacent to at most  $t-1$  vertices in  $D_2(P)$ , it follows that  $n_2 \geq |V(P)| - 2 - (t-1) = n - 2t$ .

Since  $2 \leq t \leq \frac{n-1}{3}$ , we have  $t-1 \leq \lfloor \frac{n-t}{2} \rfloor$ . As  $T_{n,t}$  is a tree, the remaining  $t-1$  vertices in  $V(T_{n,t}) - V(P)$  are adjacent to at least one vertex in  $D_2(P) \cap D_{\geq 3}(G)$  and contains at least one vertex in  $D_1(G) - V(P)$ , it follows that  $n_2 \leq n - N_3 - n_1 \leq n - 4$ . We conclude that  $n - 2t \leq \alpha_{k-reg}(T_n) \leq n - 4$ .

(ii) Suppose  $\frac{n}{3} \leq t \leq n - 5$ . We first assume  $n_1 \geq \frac{n+2}{3}$ . Since  $D_1(T_{n,t})$  is a  $k$ -independent set, it follows that  $\alpha_{k-reg}(T_{n,t}) \geq n_1 \geq \frac{n+2}{3}$ . Next, we assume  $n_1 < \frac{n+2}{3}$ , and so it follows that  $n_2 \geq n - 2n_1 + 2$ ,  $n_1 \geq N_3 + 2$  and  $n_2 = n - n_1 - N_3$ . As  $D_2(G)$  is a  $k$ -independent set, it follows that  $\alpha_{k-reg}(T_{n,t}) \geq n_2 \geq n - 2n_1 + 2 \geq n - \frac{2n+4}{3} + 2 = \frac{n+2}{3}$ . The upper bound follows from (3.9).

(iii) Suppose that  $t = n - 4$ . Then  $\text{diam}(T_{n,t}) = n - t = 4$  and  $h = 2$ . By (3.3),  $f(n, t) = \lceil \frac{n-1}{2} \rceil$ . Since vertices in  $D_2(T)$  cannot be the end vertices of  $P$  and of  $P_w$ , for each  $w \in D_1(T) - \{v_1, v_{n-t+1}\}$ , and cannot be in  $N_3$ , it follows that

$$\begin{aligned} n_2 &\leq |V(P) - \{v_1, v_{j_0}, v_{n-t+1}\}| + \sum_{w \in D_1(T) - \{v_1, v_{n-t+1}\}} (|V(P_w)| - 2) \\ &= (n - t + 1) - 3 + (n_1 - 2)(h - 1) \end{aligned} \quad (3.10)$$

Since  $n - t = 4$  and  $h = 2$ , (3.10) leads to  $n_2 \leq n_1$ . By Lemma 3.4,  $\alpha_{k-reg}(T_{n,t}) = n_1$ . By Lemma 3.3,  $t+1 \geq n_1 \geq f(n, t) = \lceil \frac{n-1}{2} \rceil$ . Thus (iii) must hold.

(iv) Suppose that  $t = n - 3$ . Then  $\text{diam}(T_{n,t}) = n - t = 3$  and  $h = 1$ . Thus by (3.10),  $n_2 < n_1$  and so by Lemma 3.4,  $\alpha_{k-reg}(T_{n,t}) = n_1$ . By Lemma 3.3,  $t+1 \geq n_1 \geq f(n, t)$ . By (3.3) with  $n - t = 3$ , we have  $f(n, t) = t+1$ . This implies (iv).

■

The bounds in Theorem 3.5 are best possible in some sense, as can be seen in the following examples.

**Example 1:** (1) Let  $P = v_1 \cdots v_{n-t+1}$  be a path. For the lower bound, let  $v'_1, v'_2, \dots, v'_{t-1}$  be vertices not in  $V(P)$  with  $2 \leq t \leq \frac{n-1}{3}$ . Since  $n \geq 2t$ , there exists distinct vertices  $v_{i_1}, v_{i_2}, \dots, v_{i_{t-1}} \in V(P) - \{v_1, v_{n-t+1}\}$ . Obtain a  $T_{n,t}$  with  $V(T_{n,t}) = V(P) \cup \{v'_1, v'_2, \dots, v'_{t-1}\}$  and  $E(T_{n,t}) = E(P) \cup \{v_{i_j} v'_j : 1 \leq j \leq t-1\}$ , (see Figure 1 (a)). Then in this  $T_{n,t}$ , we have  $n_1 = t+1$ ,  $n_2 = n-2t$ ,  $N_3 = t-1$ . Thus any  $k$ -regular independent set  $W$  must be a subset of  $D_j(G)$ , for some  $j$  with  $1 \leq j \leq 3$ . Since  $n \geq 3t+1$ , we have  $n_2 \geq n_1$ . As  $k \geq 2$ ,  $D_2(G)$  is a maximum regular  $k$ -independent set of  $T_{n,t}$ , and so  $\alpha_{k-reg}(T_{n,t}) = n-2t$  if  $n \geq 3t+1$ .

For the upper bound, let  $L = v'_0 v'_1 v'_2 \cdots v'_{t-1}$  denote a path. Obtain a  $T'_{n,t}$  from  $P$  and  $T$  by identifying the vertex  $v_j \in V(P)$  and  $v'_0 \in V(L)$ , where  $j = \lfloor \frac{n-t}{2} \rfloor + 1$ , (see Figure 1 (b)). In this case, we have  $n_1 = 3$ ,  $n_2 = n-4$ ,  $N_3 = 1$ , and so when  $k \geq 2$  and  $n \geq 7$ ,  $\alpha_{k-reg}(T'_{n,t}) = n-4$ , which shows the upper bound is sharp.

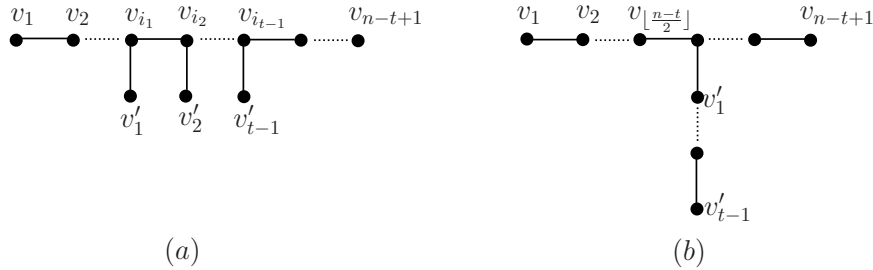


Figure 1. (a) Tree with  $\alpha_{k-reg}(T) = n-2t$  for  $k \geq 2$ . (b) Tree with  $\alpha_{k-reg}(T) = n-4$  for  $k \geq 2$ .

**Example 2:** (2) For the lower bound, we let  $n = 3t-2$  for some integer  $t \geq 4$ . Then  $n-t+1 = 2t-1$ . Let  $P = v_1 v_2 \cdots v_{2t-1}$  be a path. let  $v'_1, v'_2, \dots, v'_{t-1}$  be vertices not in  $V(P)$ . Obtain a  $T_{n,t}(1)$  with  $V(T_{n,t}(1)) = V(P) \cup \{v'_1, v'_2, \dots, v'_{t-1}\}$  and  $E(T_{n,t}(1)) = E(P) \cup \{v_{j+1} v'_j : 1 \leq j \leq t-2\} \cup \{v'_{t-2} v'_{t-1}\}$ , (see Figure 2 (a)). Then in this  $T_{n,t}(1)$ , we have  $n_1 = t = n_2$ ,  $N_3 = t-2$ . As  $k \geq 2$ , each of  $D_1(T_{n,t}(1))$  and  $D_2(T_{n,t}(1))$  is a maximum  $k$ -regular independent set, and so  $\alpha_{k-reg}(T_{n,t}(1)) = t = \frac{n+2}{3}$ .

For the upper bound, let  $n = 12$  and  $t = 4$ , and let  $T_{n,t}(2)$  be the tree depicted in Figure 2(b). Then we have  $n_1 = 3$ ,  $n_2 = 8$ ,  $N_3 = 1$ . It is routine to see that  $\alpha_{k-reg}(T_{n,t}(2)) = n - f(n, t) - 1 = 8$  for  $k \geq 2$ . Let  $n = 11$  and  $t = 4$ , and let  $T_{n,t}(3)$  be the tree depicted in Figure 2(c). Then we have  $n_1 = 3$ ,  $n_2 = 7$ ,  $N_3 = 1$  and  $\alpha_{k-reg}(T_{n,t}(3)) = n - f(n, t) - 1 = 7$  for  $k \geq 2$ . Let  $n = 9$  and  $t = 4$ , and let  $T_{n,t}(4)$  be the tree depicted in Figure 2(d). Then we have  $n_1 = 5$ ,  $n_2 = 1$ ,  $N_3 = 3$  and  $\alpha_{k-reg}(T_{n,t}(4)) = t+1 = 5$  for  $k \geq 2$ .

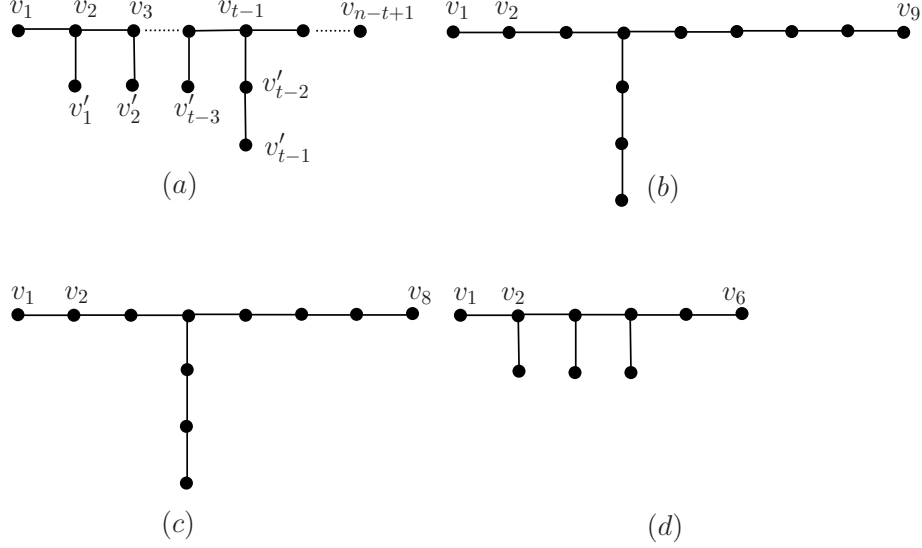


Figure 2. (a) Tree with  $\alpha_{k-reg}(T) = \frac{n+2}{3}$  for  $k \geq 2$ .  
(b) Tree with  $\alpha_{k-reg}(T) = n - f(n, t) - 1 = 8$  for  $k \geq 2$ .  
(c) Tree with  $\alpha_{k-reg}(T) = n - f(n, t) - 1 = 7$  for  $k \geq 2$ .  
(d) Tree with  $\alpha_{k-reg}(T) = t + 1 = 5$  for  $k \geq 2$ .

**Example 3:** (3) For the lower bound, let  $n = 10$  and  $t = 6$ , and let  $T_{n,t}(5)$  be the tree depicted in Figure 3(a). Then we have  $n_1 = 5$ ,  $n_2 = 4$ ,  $N_3 = 1$ , and  $\alpha_{k-reg}(T_{n,t}(5)) = \lceil \frac{n-1}{2} \rceil = 5$  for  $k \geq 2$ . For the upper bound, let  $n = 10$  and  $t = 6$ , and let  $T_{n,t}(6)$  be the tree depicted in Figure 3(b). Then we have  $n_1 = 7$ ,  $n_2 = 2$ ,  $N_3 = 1$  and  $\alpha_{k-reg}(T_{n,t}(6)) = t + 1 = 7$  for  $k \geq 2$ .

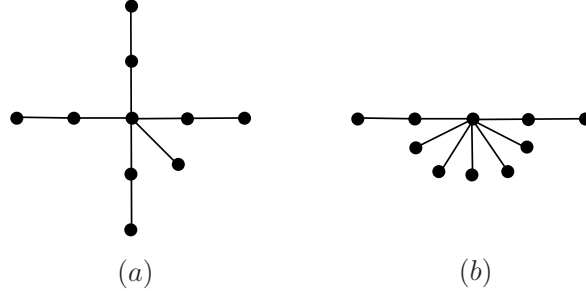


Figure 3.(a) Tree with  $\alpha_{k-reg}(T) = \lceil \frac{n-1}{2} \rceil = 5$  for  $k \geq 2$ .  
(b) Tree with  $\alpha_{k-reg}(T) = t + 1 = 7$  for  $k \geq 2$ .

## 4 Results for line graphs

In this section, we investigate the bounds for the regular  $k$ -independence number of line graphs of graphs in certain families, including trees, maximal outerplanar graphs

and triangulations. Throughout this section,  $G$  denotes a graph with  $m = |E(G)|$ , and define  $\delta' = \delta(L(G))$  and  $\Delta' = \Delta(L(G))$ . For each integer  $i$  with  $\delta' \leq i \leq \Delta'$ , define  $\Gamma_i = L(G)[D_i(L(G))]$ . Recall that the repetition number of a graph  $G$ , defined in (1.2), is the maximum  $|D_i(G)|$  over all possible values of  $i$ .

**Lemma 4.1** (*Caro and Wes [8]*) *Let  $G$  be a graph with  $m$  edges. Then*

$$\text{rep}(L(G)) \geq \frac{1}{4}m^{\frac{1}{3}}.$$

**Theorem 4.2** *Let  $G$  be a graph with  $m$  edges. Then*

$$\alpha_{k\text{-reg}}(L(G)) \geq \frac{m^{\frac{1}{3}}}{4\chi_k(L(G))}.$$

*Proof.* Since  $\alpha_{k,j}(L(G)) \geq \frac{V(\Gamma_j)}{\chi_k(\Gamma_j)}$  holds for every  $j$ , we have

$$\alpha_{k\text{-reg}}(L(G)) = \max \{ \alpha_{k,j}(L(G)), \delta' \leq j \leq \Delta' \} \geq \max \left\{ \frac{V(\Gamma_j)}{\chi_k(\Gamma_j)} : \delta' \leq j \leq \Delta' \right\}. \quad (4.11)$$

Since  $\chi_k(L(G)) \geq \chi_k(L(G_j))$  holds for every  $j$ , it follows that

$$\begin{aligned} \max_{\delta' \leq j \leq \Delta'} \left\{ \frac{V(\Gamma_j)}{\chi_k(\Gamma_j)} \right\} &\geq \max_{\delta' \leq j \leq \Delta'} \left\{ \frac{\text{rep}(L(G))}{\chi_k(\Gamma_j)} : |V(\Gamma_j)| = \text{rep}(L(G))\delta' \leq j \leq \Delta' \right\} \\ &\geq \frac{\text{rep}(L(G))}{\chi_k(L(G))}. \end{aligned} \quad (4.12)$$

By Lemma 4.1, we have  $\text{rep}(L(G)) \geq \frac{1}{4}m^{\frac{1}{3}}$ , and so  $\alpha_{k\text{-reg}}(L(G)) \geq \frac{m^{\frac{1}{3}}}{4\chi_k(L(G))}$ . ■

**Lemma 4.3** (*Caro and Wes [8]*) *Let  $G$  be a graph with average degree  $d$ , minimum degree  $\delta$ , and  $m$  edges. If  $d \geq \delta \geq 1$ , then*

$$\text{rep}(L(G)) \geq \alpha\sqrt{m} - 1,$$

where  $\alpha = \frac{\delta}{\sqrt{cd(cd-\delta)}}$  with  $c = 2d - 2\delta + 1$ .

**Theorem 4.4** *Let  $G$  be a graph with average degree  $d$ , minimum degree  $\delta$ , and  $m$  edges. If  $d \geq \delta \geq 1$ , then*

$$\alpha_{k\text{-reg}}(L(G)) \geq \frac{\alpha\sqrt{m} - 1}{\chi_k(L(G))},$$

where  $\alpha = \frac{\delta}{\sqrt{cd(cd-\delta)}}$  with  $c = 2d - 2\delta + 1$ .

*Proof.*

Since for every  $j$ , we have  $\alpha_{k,j}(L(G)) \geq \frac{V(\Gamma_j)}{\chi_k(\Gamma_j)}$ , it follows that (4.11) must hold. Hence, since for every  $j$ , we have  $\chi_k(L(G)) \geq \chi_k(\Gamma_j)$ , (4.12) also holds. By Lemma 4.3,  $\text{rep}(L(G)) \geq \alpha\sqrt{m} - 1$ , where  $\alpha = \frac{\delta}{\sqrt{cd(cd-\delta)}}$  with  $c = 2d - 2\delta + 1$ . It follows that  $\alpha_{k-\text{reg}}(L(G)) \geq \frac{\alpha\sqrt{m}-1}{\chi_k(L(G))}$ .  $\blacksquare$

**Lemma 4.5** (Caro and Wes [8]) *For a tree or maximal planar graph with  $m$  edges, the repetition number of the line graph is at least  $\sqrt{\frac{m}{30}}$  or  $\sqrt{\frac{m}{182}}$ , respectively.*

**Corollary 4.6** (1) *If  $G$  is a tree, then*

$$\alpha_{k-\text{reg}}(L(G)) \geq \frac{\sqrt{m}}{\sqrt{30}\chi_k(L(G))}.$$

(2) *If  $G$  is a maximal planar graph with  $m$  edges, then*

$$\alpha_{k-\text{reg}}(L(G)) \geq \frac{\sqrt{m}}{\sqrt{182}\chi_k(L(G))}.$$

*Proof.* Let  $n = |V(G)|$ . If  $G$  is a tree, then  $\delta = 1$ ,  $d = 2 - \frac{2}{n}$ , and  $c = 3 - \frac{4}{n}$ . It follows from Theorem 4.4 that  $\alpha_{k-\text{reg}}(L(G)) \geq \frac{\sqrt{m}}{\sqrt{30}\chi_k(L(G))}$ . If  $G$  is a maximal planar graph, then  $\delta = 3$ ,  $d = 6 - \frac{12}{n}$ , and  $c = 7 - \frac{12}{n}$ . It follows from Theorem 4.4 that  $\alpha_{k-\text{reg}}(L(G)) \geq \frac{\sqrt{m}}{\sqrt{182}\chi_k(L(G))}$ .  $\blacksquare$

**Lemma 4.7** (Caro and Wes [8]) *Let  $G$  be a graph with  $m$  edges. If  $G$  is a tree with a perfect matching, a maximal outerplanar graph, or a triangulation with a 2-factor, then  $\text{rep}(L(G))$  is at least  $\frac{m}{6}$ ,  $\frac{m}{14}$ , or  $\frac{m}{33}$ , respectively. The lower bound improves to  $\frac{m}{27}$  or  $\frac{m}{15}$  for triangulations having a 2-factor and minimum degree 4 or 5, respectively.*

**Theorem 4.8** (1) *If  $G$  is a tree with a perfect matching, then*

$$\alpha_{k-\text{reg}}(L(G)) \geq \frac{m}{6\chi_k(L(G))}.$$

(2) *If  $G$  is a maximal outerplanar graph with a 2-factor, then*

$$\alpha_{k-\text{reg}}(L(G)) \geq \frac{m}{14\chi_k(L(G))}.$$

(3) *If  $G$  is a triangulation graph with a 2-factor, then*

$$\alpha_{k-\text{reg}}(L(G)) \geq \frac{m}{33\chi_k(L(G))}.$$

(4) Moreover, if  $G$  is a triangulation graph with a 2-factor and minimum degree 4 , then

$$\alpha_{k-reg}(L(G)) \geq \frac{m}{27\chi_k(L(G))}.$$

(5) Moreover, if  $G$  is a triangulation graph with a 2-factor and minimum degree 5 , then

$$\alpha_{k-reg}(L(G)) \geq \frac{m}{15\chi_k(L(G))}.$$

*Proof.*

Since for every  $j$ , we have  $\alpha_{k,j}(L(G)) \geq \frac{V(\Gamma_j)}{\chi_k(\Gamma_j)}$ , it follows that (4.11) must hold. Hence, since for every  $j$ , we have  $\chi_k(L(G)) \geq \chi_k(\Gamma_j)$ , (4.12) also holds. By Lemma 4.7,

$$rep(L(G)) \geq \begin{cases} \frac{m}{6\chi_k(L(G))} & \text{if } G \text{ is a tree with a perfect matching} \\ \frac{m}{14\chi_k(L(G))} & \text{if } G \text{ is a maximal outerplanar graph} \\ \frac{m}{33\chi_k(L(G))} & \text{if } G \text{ is a triangulation with a 2-factor} \\ \frac{m}{27\chi_k(L(G))} & \text{if } G \text{ is a triangulation with a 2-factor with } \delta(G) \geq 4 \\ \frac{m}{15\chi_k(L(G))} & \text{if } G \text{ is a triangulation with a 2-factor with } \delta(G) \geq 5 \end{cases}.$$

Thus the conclusions of the theorem follows from (4.11) and (4.12). ■

**Lemma 4.9** (Caro and Wes [8]) *Let  $G$  be a triangulation with  $m$  edges. If  $G$  has minimum degree at least 4, then  $rep(L(G)) \geq \frac{m}{68\chi_k(L(G))}$ . If  $G$  has minimum degree at least 5, then  $rep(L(G)) \geq \frac{m}{51\chi_k(L(G))}$ .*

**Theorem 4.10** (1) *If  $G$  is a triangulation graph with  $m$  edges and minimum degree at least 4, then*

$$\alpha_{k-reg}(L(G)) \geq \frac{m}{68\chi_k(L(G))}.$$

(2) *If  $G$  is a triangulation graph with  $m$  edges and minimum degree at least 5, then*

$$\alpha_{k-reg}(L(G)) \geq \frac{m}{51\chi_k(L(G))}.$$

*Proof.*

The proof of this theorem is similar to that of Theorem 4.8, using Lemma 4.9 instead of Lemma 4.7. Therefore, it is omitted. ■

## 5 Graphs with given regular $k$ -independence number

The following proposition follows immediately from definition.

**Proposition 5.1** *Let  $G$  be a simple graph of order  $n$ . Then*

$$1 \leq \alpha_{k-reg}(G) \leq n.$$

In the rest of this section, we will present characterizations for graphs reaching either bounds in Proposition 5.1.

**Lemma 5.2** *Let  $G$  be a simple graph with  $n \geq 2$  vertices. Then there exist at least two vertices with the same degree in  $G$ .*

*Proof.* This follows from the observation that either  $G$  has an isolated vertex, and so for any  $v \in V(G)$ ,  $0 \leq d_G(v) \leq n - 2$ ; or  $G$  has no isolated vertices, and so for any  $v \in V(G)$ ,  $1 \leq d_G(v) \leq n - 1$ . ■

**Theorem 5.3** *Let  $G$  be a simple graph. Then  $\alpha_{k-reg}(G) = 1$  if and only if  $k = 0$  and any subset of vertices with same degree in  $G$  induces a clique of  $G$ .*

*Proof.* Suppose  $\alpha_{k-reg}(G) = 1$ . Clearly,  $k = 0$  and  $\alpha_{0-reg}(G) = 1$ . Then  $\alpha_{0-reg}(G) = \alpha_{reg}(G) = 1$ . By Lemma 5.2, there exist at least two vertices with same degree in  $G$ . By the definition of the regular independence number, there exists an edge between any two vertices with same degree in  $G$ . Hence, any subset of vertices with same degree in  $G$  induces a clique in  $G$ , as desired.

Conversely, suppose that  $k = 0$  and any subset of vertices with same degree in  $G$  induces a clique of  $G$ . Note that any two vertices with same degree in  $G$  are adjacent. By the definition of the regular independence number, we have  $\alpha_{reg}(G) = 1$  and  $\alpha_{0-reg}(G) = \alpha_{reg}(G)$ . So,  $\alpha_{0-reg}(G) = 1$ . ■

**Theorem 5.4** *Let  $h$  be a nonnegative integer. Then  $\alpha_{k-reg}(G) = n$  if and only if  $G$  is a  $h$ -regular graph with  $n$  vertices and  $k \geq h$ .*

*Proof.* Suppose  $\alpha_{k-reg}(G) = n$ . By the definition of the regular  $k$ -independence number, we have that all vertices in  $G$  have same degree in  $G$ . Hence, the graph  $G$  is a  $h$ -regular graph with  $n$  vertices and  $k \geq h$ .

Conversely, if  $G$  is a  $h$ -regular graph with  $n$  vertices and  $k \geq h$ , then all vertices form a regular  $k$ -independent set. By the definition of the regular  $k$ -independence number, we have  $\alpha_{k-reg}(G) = n$  for  $k \geq h$ . ■

## 6 Nordhaus-Gaddum-type results

In this section, we investigate the Nordhaus-Gaddum-type problem on the regular  $k$ -independence number of graphs and obtain the sharp bounds for both  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G})$ , and  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G})$ , over the class  $\mathcal{G}(n)$  and characterize the extremal graphs.

**Theorem 6.1** *For any  $G \in \mathcal{G}(n)$ ,  $\bar{G}$  denotes the complement of  $G$ . Then*

- (1)  $3 \leq \alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) \leq 2n$ ;
- (2)  $2 \leq \alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) \leq n^2$ .

*Proof.* (1) By Proposition 5.1,  $\alpha_{k-reg}(G) \geq 1$  and  $\alpha_{k-reg}(\bar{G}) \geq 1$ , and so  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) \geq 2$ . However, By Proposition 5.1,  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 2$  if and only if  $\alpha_{k-reg}(G) = 1$  and  $\alpha_{k-reg}(\bar{G}) = 1$ . Thus if  $\alpha_{k-reg}(G) = 1 = \alpha_{k-reg}(\bar{G})$ , then by Theorem 5.3, for any  $i$ , either  $D_i(G) = \emptyset$  or  $G[D_i(G)]$  is a clique. By the same reason, either  $D_{n-i-1}(\bar{G}) = \emptyset$  or  $\bar{G}[D_{n-i-1}(\bar{G})]$  is a clique. Since  $G \cup \bar{G} = K_n$ , it is impossible that both  $G[D_i(G)]$  and  $\bar{G}[D_{n-i-1}(\bar{G})]$  are cliques. This contradiction shows that we must have  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) \geq 3$ . By Proposition 5.1,  $\alpha_{k-reg}(G) \leq n$  and  $\alpha_{k-reg}(\bar{G}) \leq n$ , and so  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) \leq 2n$ .

(2) By Proposition 5.1,  $\alpha_{k-reg}(G) \geq 1$  and  $\alpha_{k-reg}(\bar{G}) \geq 1$ , and so  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) \geq 1$ . By Proposition 5.1,  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = 1$  if and only if  $\alpha_{k-reg}(G) = 1 = \alpha_{k-reg}(\bar{G})$ , and so we obtain a contradiction as above. Therefore,  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) \geq 2$ . From Proposition 5.1,  $\alpha_{k-reg}(G) \leq n$  and  $\alpha_{k-reg}(\bar{G}) \leq n$ , and so  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) \leq n^2$ . ■

Before we study the graphs reaching the bounds in Theorem 6.1, we make the following observations.

**Observation 6.2** *Let  $n \geq 2$  be an integer, and  $G \in \mathcal{G}(n)$ . Then the following are equivalent.*

- (i)  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 3$ .
- (ii)  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = 2$ .
- (iii)  $\{\alpha_{k-reg}(G), \alpha_{k-reg}(\bar{G})\} = \{1, 2\}$ .

In fact, by Proposition 5.1, each of (i) and (ii) of Observation 6.2 is equivalent to (iii).

**Observation 6.3** *Let  $n \geq 2$  be an integer, and  $G \in \mathcal{G}(n)$ . Then the following are equivalent.*

- (i)  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 2n$ .



- (ii)  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = n^2$ .
- (iii)  $\alpha_{k-reg}(G) = n = \alpha_{k-reg}(\bar{G})$ .

**Proposition 6.4** *Let  $n \geq 2$  be an integer, and let  $G \in \mathcal{G}(n)$ . Then*

$$\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 3 \text{ or } \alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = 2 \quad (6.13)$$

*if and only if  $G$  satisfies the following conditions.*

- (1)  $k = 0$ ;
- (2) *for any integer  $i \geq 0$ ,  $|D_i(G)| \leq 2$  and  $|D_i(\bar{G})| \leq 2$ .*
- (3) *Either every  $G[D_i(G)]$  is connected for any  $i$  such that  $D_i(G) \neq \emptyset$ , or every  $\bar{G}[D_j(\bar{G})]$  is connected for any  $j$  such that  $D_j(\bar{G}) \neq \emptyset$ .*

*Proof.* If (6.13) holds, then by Observation 6.2 and by symmetry, we assume that  $\alpha_{k-reg}(G) = 1$  and  $\alpha_{k-reg}(\bar{G}) = 2$ . By Theorem 5.3, we have  $k = 0$  (and so (1) holds), and for each  $i$  with  $|D_i(G)| \geq 2$ ,  $G[D_i(G)]$  must be a clique. If  $|D_i(G)| \geq 3$ , then  $D_i(G)$  is a regular independent set in  $\bar{G}$ , contrary to the fact that  $\alpha_{k-reg}(\bar{G}) = 2$ . This implies that

$$|D_i(G)| \leq 2, \text{ for every } i. \quad (6.14)$$

If for some  $j$ ,  $|D_j(\bar{G})| \geq 3$ , then as  $D_{n-j-1}(G) = D_j(\bar{G})$ , we have  $|D_{n-j-1}(G)| \geq 3$ , contrary to (6.14). Hence  $|D_j(\bar{G})| \leq 2$  for every  $j$ , and so (2) holds.

To justify (3), we observe that if for some  $i \neq j$ , both  $G[D_i(G)]$  and  $\bar{G}[D_j(\bar{G})]$  are disconnected, then by (2),  $G[D_i(G)]$  is independent in  $G$  and  $\bar{G}[D_j(\bar{G})]$  is independent in  $\bar{G}$ . It follows that  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) \geq |D_i(G)| + |D_j(\bar{G})| \geq 2 + 2 = 4$ , contrary to (6.13). Thus (3) must hold.

Conversely, suppose  $G$  satisfies Proposition 6.4 (1), (2) and (3). Then by  $n \geq 2$  and by (2), every regular 0-independent set in  $G$  and in  $\bar{G}$  is of size at most 2. By (3),  $\{\alpha_{k-reg}(G), \alpha_{k-reg}(\bar{G})\} = \{1, 2\}$ , and so by Observation 6.2, (6.13) must hold. ■

**Remark 1:** In fact, graphs satisfying the conditions in Proposition 6.4 indeed exist. Let  $H$  be a graph obtained from a triangle and an edge by identifying a vertex of this triangle and an endpoint of this edge. Clearly,  $|V(H)| = 4$ . Let  $G$  be a union of  $H$  and an isolated vertex. Then,  $|V(G)| = 5$ . It is routine to check that graphs  $G$  and  $\bar{G}$  satisfy  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 3$  and  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = 2$  for  $k = 0$ .

**Proposition 6.5** *Let  $n \geq 2$  and  $h \geq 0$  be integers, and let  $G \in \mathcal{G}(n)$ . The following are equivalent.*

- (i)  $\alpha_{k-reg}(G) + \alpha_{k-reg}(\bar{G}) = 2n$ .
- (ii)  $\alpha_{k-reg}(G) \cdot \alpha_{k-reg}(\bar{G}) = n^2$ .
- (iii)  $G$  is an  $h$ -regular graph with  $n$  vertices and  $k \geq \max\{h, n - 1 - h\}$ .

*Proof.* By Observation 6.3, it suffices to show that (i) and (iii) are equivalent.

Assume that (i) holds. By Observation 6.3,  $\alpha_{k-reg}(G) = n$ , and so  $V(G)$  is a regular  $k$ -independent set. Hence by Theorem 5.4,  $G$  must be  $h$ -regular graph and  $k \geq h$ . By Observation 6.3,  $\alpha_{k-reg}(\bar{G}) = n$ , and so the same argument shows that  $\bar{G}$  is  $n - 1 - h$ -regular and  $k \geq n - 1 - h$ , and so (iii) must hold.

Assume that (iii) holds. Then  $G$  is an  $h$ -regular graph of order  $n$ , and  $\bar{G}$  is  $(n - 1 - h)$ -regular. Since  $k \geq \max\{h, n - 1 - h\}$ ,  $V(G)$  is a regular  $k$ -independent set in both  $G$  and  $\bar{G}$ , and so by Theorem 5.4,  $\alpha_{k-reg}(G) = n = \alpha_{k-reg}(\bar{G}) = n$ . Thus (i) follows. ■

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